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Symmetrization of the Berezin star product and multiple star product method

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Abstract
We construct a multiple star product method and, using this method, show that integral forms of some star products can be written in terms of the path integral. The method is applied to some examples. In particular, the associativity of the skew-symmetrized Berezin star product proposed by Saito and Wakatsuki is recovered in the large-N limit of the multiple star product. We also derive the path integral form of the Kontsevich star product from the multiple Moyal star product. This paper includes a review of star products.

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1. Introduction

In recent years, the relation between superstring theory and a deformation quantization has been explored. D-branes, boundaries of open strings, are non-perturbative objects of superstring theory. Matrix models [BFSS, IKKT] were proposed as a D-brane action a few years ago. It is shown that a stable solution of this action is a noncommutative manifold [CDS, AIIKKT]. This noncommutativity, however, comes from the Moyal quantization. So, this solution implies a flat D-brane. (We have also derived a noncommutative gauge theory on a fuzzy sphere from the matrix model [IKTW].) If we regard a D-brane as space–time, we should study the deformation quantization of curved spaces in order to realize the quantum gravity.

In order to proceed further it is useful to clarify the mathematical background of the deformation quantization. Star products were first introduced by Groenewold [Gr], and are now known as Moyal products [Mo]. They associate an operator product to a noncommutative product of functions. Here, the operators are mapped into the functions by taking account of the Weyl ordering. Weyl ordering means a skew-symmetric definition as we will see later. Berezin tried to quantize curved phase spaces about 25 years ago and succeeded in quantizing some Kähler manifolds, e.g. spheres [Be]. Recently the Berezin quantization has been generalized to the arbitrary Kähler manifold [RT]. However the Berezin quantization is defined without skew symmetry, and hence is not a generalization of the Moyal one.

The correlation between these methods of quantization is, however, not at all clear. In this paper, we attempt to skew-symmetrize the Berezin quantization by means of the multiple star product method. The multiple star products reduce to the path-integral form in the large-N limit (see [Sh, Al] for the original ideas). As a result, our formulation becomes similar to the path-integral form of the Kontsevich quantization which is defined perturbatively on the Poisson manifold [Ko] but also can be described by a bosonic string path integral [CF]. In particular, in the flat case, our star product coincides with the Kontsevich star product.

The paper is structured as follows. In section 2, we review deformation quantization, in particular Moyal [Mo], Berezin [Be, MM] and Kontsevich [Ko, CF] quantization. In section 3, we construct the multiple star product method and explain the symmetrized Berezin (or Wick type) star product [SW, Ma]. We also study its associativity in detail. In section 4, we derive the path-integral form of the Kontsevich star product on the flat plane from the multiple star product method. Section 5 is devoted to discussion. The appendix gives some examples.

2. Deformation quantization

This section includes the definition and properties of deformation quantization. We also briefly review Moyal, Berezin and Kontsevich quantization as examples.

2.1. General definition and property

The deformation quantization [BFFLS, St] is provided by a star product, which is defined by

$$f * g = \sum_{m=0}^{\infty} B_m(f, g)\lambda^m \tag{1}$$

where

- λ is a deformation parameter,
- $f, g \in A = C^\infty(M)[[\lambda]]$, where $C^\infty(M)[[\lambda]]$ means that the coefficients of the λ power series are C^∞ functions on M,
- B_m are bi-differential operators ($B_m : A \times A \to A$).

The deformation quantization has the following properties:

(1) associativity

$$f * (g * h) = (f * g) * h. \tag{2}$$

(2) $m = 0$

$$B_0(f, g) = fg. \tag{3}$$

(3) $m = 1$

$$B_1(f, g) - B_1(g, f) = \{f, g\} = 2\sum_{i,j} \alpha^{ij}\partial_i f \partial_j g \tag{4}$$

where $i, j = 1, 2, \ldots, d = \dim(M)$ and $\{\cdot, \cdot\}$ is a Poisson bracket which satisfies

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h\{f, g\}\} = 0 \tag{5}$$

so the skew-symmetric bivector field α satisfies

$$\alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0. \tag{6}$$

Table 1.

Manifold	Flat plane	Kähler	Poisson
Quantization	Moyal	Berezin	Kontsevich
Symbol	$\star$	$\boxed{\star}$	$*$
$m = 1$	skew-symmetric	asymmetric	skew-symmetric

It can be shown that one or more star products determined by (2)–(4) exist. The deformation quantization has the following equivalence, called a gauge equivalence. $*$ and $*'$ are identified if

$$f' *' g' = \mathrm{D}(f * g) \tag{7}$$

where $f' = \mathrm{D}(f),\ g' = \mathrm{D}(g)$ and D is a differential operator ($\mathrm{D} : A \to A$). However, we can take two simple gauges. One is the skew-symmetric gauge

$$B_1(f, g) = \tfrac{1}{2}\{f, g\} = \sum_{i,j} \alpha^{ij} \partial_i f \partial_j g. \tag{8}$$

The other gauge is

$$B_1(f, g) = \sum_{i,j} \beta^{ij} \partial_i f \partial_j g \tag{9}$$

where β is the upper triangle matrix of α which satisfies $\alpha^{ij} = \beta^{ij} - \beta^{ji}$. So we call the star products determined by (8) and (9) the skew-symmetric and asymmetric products respectively. We have three concrete examples of deformation quantization, as shown in table 1.

From now on we consider real two-dimensional manifolds for the sake of simplicity.

2.2. *Moyal quantization*

The Poisson bracket on the flat plane is defined by

$$\{f, g\} = \sum_{i,j} \varepsilon^{ij} \partial_i f \partial_j g = \partial_x f \partial_p g - \partial_p f \partial_x g. \tag{10}$$

Thus $\alpha^{ij} = \varepsilon^{ij}/2$ by (4). So we obtain the associative star product on the flat plane, i.e. the Moyal star product, as

$$\begin{aligned} f \star g(x, p) &= f(x, p)\mathrm{e}^{\frac{\lambda}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x})} g(x, p) \\ &= fg + \lambda \tfrac{1}{2}\{f, g\} + \mathrm{O}(\lambda^2). \end{aligned} \tag{11}$$

This star product agrees with equations (3) and (8), and satisfies the associativity from the following results:

$$e_1 \star (e_2 \star e_3) = \mathrm{e}^{-\frac{\lambda}{2}(m_1n_2+m_2n_3+n_3m_1-n_1m_2-n_2m_3-m_3n_1)} e_1e_2e_3 = (e_1 \star e_2) \star e_3 \tag{12}$$

where the e_i are the Fourier series

$$e_i = \mathrm{e}^{\mathrm{i}(m_i x + n_i p)}.$$

Here we require the usual canonical commutation relation

$$[x, p]_\star = x \star p - p \star x = \mathrm{i}\hbar \tag{13}$$

so that we obtain $\lambda = \mathrm{i}\hbar$. This star product can also be written in the integral form [Ba], because we have the following relations:

$$\mathrm{e}^{\mathrm{i}(mx+np)} \star \mathrm{e}^{\mathrm{i}(m'x+n'p)} = \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}(mn'-m'n)} \mathrm{e}^{\mathrm{i}(mx+np)} \mathrm{e}^{\mathrm{i}(m'x+n'p)} \tag{14}$$

and

$$\int \frac{\mathrm{d}w\,\mathrm{d}\eta}{\pi\hbar}\frac{\mathrm{d}w'\,\mathrm{d}\eta'}{\pi\hbar}\mathrm{e}^{\frac{2\mathrm{i}}{\hbar}S}\mathrm{e}^{\mathrm{i}(mw+n\eta)}\mathrm{e}^{\mathrm{i}(m'w'+n'\eta')} = \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}(mn'-m'n)}\mathrm{e}^{\mathrm{i}(mx+np)}\mathrm{e}^{\mathrm{i}(m'x+n'p)} \tag{15}$$

where

$$S = \begin{vmatrix} 1 & 1 & 1 \\ x & w & w' \\ p & \eta & \eta' \end{vmatrix}.$$

The left-hand sides of equations (14) and (15) are equivalent, so that we obtain the integral form of the Moyal star product after multiplying arbitrary Fourier coefficients and integrating over m, n as

$$f \star g(x, p) = \int \frac{\mathrm{d}w\,\mathrm{d}\eta}{\pi\hbar}\frac{\mathrm{d}w'\,\mathrm{d}\eta'}{\pi\hbar}\mathrm{e}^{\frac{2\mathrm{i}}{\hbar}S} f(w, \eta) g(w', \eta'). \tag{16}$$

2.3. Berezin quantization

The Poisson bracket on the Kähler manifold is given as

$$\{f, g\} = \frac{2}{\mathrm{i}} h^{z\bar{z}}(\partial_z f \partial_{\bar{z}} g - \partial_{\bar{z}} f \partial_z g) \tag{17}$$

where $h^{z\bar{z}}$ is the inverse of a Kähler metric

$$h_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z, \bar{z})$$

and $K(z, \bar{z})$ is a Kähler potential. The factor $\frac{2}{\mathrm{i}}$ in equation (17) is necessary in order that the Poisson bracket becomes equation (10) in the flat case. The original Berezin quantization covers only the Ricci flat Kähler manifold. The Berezin star product is defined by

$$f \circledast g(z, \bar{z}) = \int \mathrm{d}\mu_\nu(v, \bar{v}) \mathrm{e}^{\frac{1}{\nu}\Phi(z, \bar{z}, v, \bar{v})} f(z, \bar{v}) g(v, \bar{z}) \tag{18}$$

where $\Phi(z, \bar{z}, v, \bar{v})$ is called the Calabi function and defined by the Kähler potential as

$$\Phi(z, \bar{z}, v, \bar{v}) = K(z, \bar{v}) + K(v, \bar{z}) - K(z, \bar{z}) - K(v, \bar{v}) \tag{19}$$

and the measure $\mathrm{d}\mu_\nu$ is determined by the metric as

$$\mathrm{d}\mu_\nu(z, \bar{z}) = h_{z\bar{z}} \frac{\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\pi\nu}. \tag{20}$$

This star product can be expanded around $\lambda = \nu/2\mathrm{i} = 0$ as follows:

$$f \circledast g(z, \bar{z}) = fg + \lambda(B_1^+(f, g) + B_1^-(f, g)) + \mathrm{O}(\lambda^2) \tag{21}$$

where B_1^+ and B_1^- are a symmetric part and a skew-symmetric part respectively,

$$B_1^+ = 2\mathrm{i}Afg - \tfrac{1}{2}\{f, g\}_+ \qquad B_1^- = \tfrac{1}{2}\{f, g\} \tag{22}$$

and

$$A = \tfrac{1}{2} h^{z\bar{z}} \partial_z \partial_{\bar{z}} \log h_{z\bar{z}} \qquad \{f, g\}_+ = \frac{2}{\mathrm{i}} h^{z\bar{z}}(\partial_z f \partial_{\bar{z}} g + \partial_{\bar{z}} f \partial_z g).$$

If the manifold M is the Kähler manifold, $A = 0$ [RT]. Thus (3) and (4) are satisfied. The associativity is written as

$$((f \circledast g) \circledast h)(z, \bar{z}) = \int \mathrm{d}\mu_\nu(v, \bar{v})\,\mathrm{d}\mu_\nu(u, \bar{u}) f(z, \bar{v}) g(v, \bar{u}) h(u, \bar{z}) \mathrm{e}^{\frac{1}{\nu}(\Phi(z, \bar{u}, v, \bar{v}) + \Phi(z, \bar{z}, u, \bar{u}))}$$

$$(f \circledast (g \circledast h))(z, \bar{z}) = \int \mathrm{d}\mu_\nu(v, \bar{v})\,\mathrm{d}\mu_\nu(u, \bar{u}) f(z, \bar{v}) g(v, \bar{u}) h(u, \bar{z}) \mathrm{e}^{\frac{1}{\nu}(\Phi(v, \bar{z}, u, \bar{u}) + \Phi(z, \bar{z}, v, \bar{v}))}.$$

The Calabi function clearly satisfies

$$\Phi(z, \bar{u}, v, \bar{v}) + \Phi(z, \bar{z}, u, \bar{u}) = \Phi(v, \bar{z}, u, \bar{u}) + \Phi(z, \bar{z}, v, \bar{v})$$

so the associativity is

$$((f \circledast g) \circledast h)(z, \bar{z}) = (f \circledast (g \circledast h))(z, \bar{z}). \tag{23}$$

2.4. Kontsevich quantization

The Kontsevich quantization covers the Poisson manifold (M) which is a general manifold with the Poisson structure. Kontsevich perturbatively solved $B_m(f, g)$ under the conditions (2), (3) and (8) as

$$B_m(f, g) = \sum_{\Gamma \in G_m} w_\Gamma B_\Gamma(f, g) \tag{24}$$

where G_m is a set of diagrams related to the number m, $B_\Gamma(f, g)$ is a bi-differential operator determined by the Feynman diagram and ω_Γ is a weight [Ko]. Thus Kontsevich defines the star product on the Poisson manifold by a formal power series of λ as

$$f * g = \sum_{m=0}^{\infty} \lambda^m \sum_{\Gamma \in G_m} w_\Gamma B_\Gamma(f, g). \tag{25}$$

Cattaneo and Felder have also shown that the Kontsevich star product (25) coincides with the path-integral form of a topological bosonic string (nonlinear sigma model):

$$f * g(x) = \int_{X(\infty)=x} f(X(1))g(X(0))\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[X,\eta]}\mathcal{D}X\mathcal{D}\eta \tag{26}$$

where the action is defined on a disc D as

$$S[X, \eta] = \int_D \eta_i(u) \wedge \mathrm{d}X^i(u) + \tfrac{1}{2}\alpha^{ij}(X(u))\eta_i(u) \wedge \eta_j(u)$$

and

- $D = \{u \in R^2\ ,\ |u| \leqslant 1\}$,
- X and η are real bosonic fields,
- $X : D \to M$,
- η is a differential 1-form on $D :\ X^*(T^*M) \otimes T^*D$.

In the symplectic case, the action can be integrated over η and becomes a boundary integration by the Stokes theorem as

$$f *_{\mathrm{symp}} g(x) = \int_{\gamma(\pm\infty)=x} f(\gamma(1))g(\gamma(0))\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_\gamma \mathrm{d}^{-1}\omega}\,\mathrm{d}\gamma \tag{27}$$

where γ is a loop trajectory from x to x.

3. Symmetrized Berezin star product

In this section, we first explain the multiple star product method. Next we define the S-star product to clarify the relationship between Moyal and Berezin quantization. However, the S-star product is not associative in curved space. So, using the multiple star product method, we derive an associative star product, i.e. the O-star product.

3.1. Multiple star product method

Generally, the integral forms of the star products can be written as

$$f \diamond g(\alpha) = \int \mathrm{d}\mu_\lambda(\beta, \gamma)\ \mathrm{e}^{\mathcal{K}_\lambda(\alpha,\beta,\gamma)} f(\beta)g(\gamma) \tag{28}$$

where $\alpha, \beta, \gamma \in M$, $\mathcal{K}_\lambda = \mathcal{K}/\lambda$ is an integral kernel and $d\mu_\lambda = d\mu/\lambda^2$ is a measure which relates two points on M. We assume that this star product $\diamond$ satisfies the following:

$$f \diamond g = fg + \lambda \frac{\{f, g\}}{2} + O(\lambda^2) \tag{29}$$

$$f \diamond 1 = 1 \diamond f = f \tag{30}$$

$$d\mu_\lambda(\beta, \gamma) = d\mu_\lambda(\gamma, \beta). \tag{31}$$

We also add an assumption $\mathcal{K}_\lambda(\alpha, \beta, \beta) = 0$. Note that we do not require this star product $\diamond$ to be associative. We call this product $\diamond$ the *non-associative* star product.

Next we define the multiple star product of $\diamond$ as

$$A^N(f) = f_{N/N} \diamond f_{N-1/N} \diamond \cdots \diamond f_{2/N} \diamond f_{1/N}. \tag{32}$$

An equivalence of the forward product $\overleftarrow{A}^N$ and the backward product $\overrightarrow{A}^N$ is necessary at least in order that A^N is well defined where

$$\overleftarrow{A}^N(f) := (f_{N/N} \diamond (f_{N-1/N} \diamond (\cdots \diamond (f_{1/N} \diamond 1) \cdots)))$$
$$= \int \left(\prod_{i=1}^{N} d\mu_{\lambda N}(\beta_{i/N}, \gamma_{i/N}) f_{i/N}(\beta_{i/N}) \right) \exp \sum_{i=1}^{N} \frac{1}{N} \mathcal{K}_\lambda(\gamma_{i+1/N}, \beta_{i/N}, \gamma_{i/N}) \tag{33}$$

$$\overrightarrow{A}^N(f) := (((\cdots (1 \diamond f_{N/N}) \diamond \cdots) \diamond f_{2/N}) \diamond f_{1/N})$$
$$= \int \left(\prod_{i=1}^{N} d\mu_{\lambda N}(\beta_{i/N}, \gamma_{i/N}) f_{i/N}(\beta_{i/N}) \right) \exp \sum_{i=1}^{N} \frac{1}{N} \mathcal{K}_\lambda(\gamma_{i-1/N}, \gamma_{i/N}, \beta_{i/N}) \tag{34}$$

$$\alpha = \beta_0 = \beta_{N+1/N} = \gamma_0 = \gamma_{N+1/N}. \tag{35}$$

Note that we change the deformation parameter λ to λN. From this equivalence, we obtain a condition

$$\frac{1}{N} \sum_{i=1}^{N} (\mathcal{K}_\lambda(\gamma_{i+1/N}, \beta_{i/N}, \gamma_{i/N}) - \mathcal{K}_\lambda(\gamma_{i-1/N}, \gamma_{i/N}, \beta_{i/N})) = 0. \tag{36}$$

Using the boundary condition (35) and the additional condition $\mathcal{K}(\alpha, \beta, \beta) = 0$, this condition (36) is also deformed as

$$\frac{1}{N} \sum_{i=0}^{N} (\mathcal{K}_\lambda(\gamma_{i+1/N}, \beta_{i/N}, \gamma_{i/N}) - \mathcal{K}_\lambda(\gamma_{i/N}, \gamma_{i+1/N}, \beta_{i+1/N})) = 0. \tag{37}$$

This condition corresponds to the associativity condition in the case of $f_{i/N} = 1$ except for three $f_{i/N}$. We denote $A^N(f) := \overleftarrow{A}^N(f) = \overrightarrow{A}^N(f)$ when $\mathcal{K}_\lambda$ satisfies equation (37).

3.2. Relationship between the Moyal and Berezin star products

The Berezin star product in the flat case coincides with the Moyal one except for skew-symmetry or asymmetry. This difference is explained as follows. First, we write the Moyal star product in complex variables to make clear the correspondence with the Berezin star product:

$$f \star g(z, \bar{z}) = f(z, \bar{z}) e^{\hbar(\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}} - \overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z})} g(z, \bar{z}) \tag{38}$$

where $z = x + ip$. This star product is gauge equivalent (7) to $\star_{st}$ and $\star_{ar}$ where

$$\star_{st} = e^{2\hbar \overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}} \qquad \text{and} \qquad \star_{ar} = e^{-2\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z}} \tag{39}$$

because the gauge equivalent condition is satisfied in the case of

$$D = \mathrm{e}^{\hbar \overleftarrow{\partial_z} \overrightarrow{\partial_{\bar{z}}}} \qquad \text{and} \qquad D = \mathrm{e}^{-\hbar \overleftarrow{\partial_{\bar{z}}} \overrightarrow{\partial_z}} \tag{40}$$

respectively [Vo, Be, APS]. Thus we obtain a star product relation,

$$\star = (\star_{\mathrm{st}} \star_{\mathrm{ar}})^{\frac{1}{2}}. \tag{41}$$

Here $\star_{\mathrm{st}}^{\frac{1}{2}}$ and $\star_{\mathrm{ar}}^{\frac{1}{2}}$ can be written in integral form [APS] as

$$\begin{aligned} f \star_{\mathrm{st}}^{\frac{1}{2}} g(z, \bar{z}) &= \int \frac{\mathrm{i}\,\mathrm{d}w \wedge \mathrm{d}\bar{w}}{2\pi\theta} \mathrm{e}^{\frac{1}{\theta}|w-z|^2} f(w, \bar{z}) g(z, \bar{w}) \\ f \star_{\mathrm{ar}}^{\frac{1}{2}} g(z, \bar{z}) &= \int \frac{\mathrm{i}\,\mathrm{d}v \wedge \mathrm{d}\bar{v}}{2\pi(-\theta)} \mathrm{e}^{-\frac{1}{\theta}|v-z|^2} f(z, \bar{v}) g(v, \bar{z}) \end{aligned} \tag{42}$$

where $\theta = -\hbar$. Thus we obtain

$$f \star g(z, \bar{z}) = -\int \frac{\mathrm{i}\,\mathrm{d}v \wedge \mathrm{d}\bar{v}}{2\pi\theta} \frac{\mathrm{i}\,\mathrm{d}w \wedge \mathrm{d}\bar{w}}{2\pi\theta} \mathrm{e}^{\frac{1}{\theta}(-|v-z|^2+|w-z|^2)} f(w, \bar{v}) g(v, \bar{w}). \tag{43}$$

The star products (42) are two types of the Berezin star product on the flat plane, i.e. the term $-|v-z|^2$ is the Calabi function on the flat plane. Taking this result into account, we generalize the Moyal star product $\star$ to the S-star product on the Ricci flat Kähler manifold ([Ma] argues that the S-star product may be available to the general Kähler manifold), which is defined by

$$\begin{aligned} f \boxed{\star} g(z, \bar{z}) := \int & \mathrm{d}\mu_\theta(v, \bar{v})\, \mathrm{d}\mu_{-\theta}(w, \bar{w}) \\ & \times \exp \frac{1}{\theta}(\Phi(z, \bar{z}; v, \bar{v}) - \Phi(z, \bar{z}; w, \bar{w})) f(w, \bar{v}) g(v, \bar{w}) \end{aligned} \tag{44}$$

where $\mathrm{d}\mu_\theta(z, \bar{z}) = h_{z\bar{z}} \mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}/2\pi\theta$, similarly to (20). However, this star product is not associative unless flat. This complication is overcome by using the multiple star product method.

3.3. *Associativity of symmetrized Berezin star product*

In this section, we attempt to recover the associativity of the S-star product. First, we show that the S-star product is a non-associative star product. In the case of the S-star product, we know the following correspondence:

$$\alpha = (z, \bar{z}) \qquad \beta = (w, \bar{v}) \qquad \gamma = (v, \bar{w}) \tag{45}$$

$$\lambda = \theta \qquad \mathrm{d}\mu_\lambda(\beta, \gamma) = \mathrm{d}\mu_\theta(v, \bar{v})\, \mathrm{d}\mu_{-\theta}(w, \bar{w}) \tag{46}$$

$$\mathcal{K}_\lambda(\alpha, \beta, \gamma) = \frac{1}{\theta}(\Phi(z, \bar{z}; v, \bar{v}) - \Phi(z, \bar{z}; w, \bar{w})). \tag{47}$$

Thus the S-star product clearly satisfies the conditions (31) and $\mathcal{K}_\lambda(\alpha, \beta, \beta) = 0$. Also, it is shown in [Ma] that this product satisfies the conditions (29) and (30).

Next, we assess whether the S-star product satisfies condition (37) or not. In the S-star product, the left-hand side of (37) is written in terms of the Kähler potential K as

$$\begin{aligned} \mathrm{lhs} = \frac{1}{\theta} \frac{1}{N} \sum_{i=0}^{N} & (K(v_{i+1/N}, \bar{v}_{i/N}) + K(v_{i/N}, \bar{v}_{i+1/N}) - 2K(v_{i/N}, \bar{v}_{i/N})) \\ & -(K(w_{i+1/N}, \bar{w}_{i/N}) + K(w_{i/N}, \bar{w}_{i+1/N}) - 2K(w_{i/N}, \bar{w}_{i/N})). \end{aligned} \tag{48}$$

This result is non-zero but becomes zero in the large-N limit as

$$\mathrm{lhs} \to \frac{1}{\theta} \int_0^1 \mathrm{d}\tau\, \mathrm{d}(K(v, \bar{v}) - K(w, \bar{w})) = 0 \tag{49}$$

where $v, w = v(\tau), w(\tau)$ and the boundary conditions (35) become

$$\begin{aligned} v(0) = v(1) = w(0) = w(1) = z \\ \bar{v}(0) = \bar{v}(1) = \bar{w}(0) = \bar{w}(1) = \bar{z}. \end{aligned} \tag{50}$$

Thus $A^N(f)$ is ill-defined but $A(f) = \lim_{N\to\infty} A^N(f)$ is well defined. We call this star product a *pseudo-associative* product. By using $A(f)$ and

$$f_{i/N} = \begin{cases} g & (i/N = i_1/N \to \tau_1) \\ f & (i/N = i_2/N \to \tau_2) \\ 1 & (i/N \to \tau \neq \tau_1, \tau_2) \end{cases} \tag{51}$$

we construct an associative star product $\circledast$ in terms of the path integral form as

$$\begin{aligned} f \circledast g(z, \bar{z}) &:= A(f) \\ &= \cdots 1 \boxed{\star} 1 \boxed{\star} f \boxed{\star} 1 \boxed{\star} 1 \cdots 1 \boxed{\star} 1 \boxed{\star} g \boxed{\star} 1 \boxed{\star} 1 \cdots \\ &= \lim_{N\to\infty} \int \prod_{i=1}^{N} \mathrm{d}\mu_{\theta N}(v_{i/N}, \bar{v}_{i/N})\, \mathrm{d}\mu_{-\theta N}(w_{i/N}, \bar{w}_{i/N}) \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\theta} S_i\right] f(v_{i_2/N}, \bar{w}_{i_2/N}) g(v_{i_1/N}, \bar{w}_{i_1/N}) \\ &= \int \mathcal{D}\mu_\theta(v, \bar{v}) \mathcal{D}\mu_{-\theta}(w, \bar{w}) \exp\left[\frac{\mathrm{i}}{\theta} S\right] f(v(\tau_2), \bar{w}(\tau_2)) g(v(\tau_1), \bar{w}(\tau_1)) \end{aligned} \tag{52}$$

where actions are written as follows:

$$S_i = \frac{1}{\mathrm{i}} \frac{1}{N} \sum_{i=1}^{N} (\Phi(v_{i+1/N}, \bar{w}_{i+1/N}; v_{i/N}, \bar{v}_{i/N}) - \Phi(v_{i+1/N}, \bar{w}_{i+1/N}; w_{i/N}, \bar{w}_{i/N})) \tag{53}$$

$$S = \frac{1}{\mathrm{i}} \int_{-\infty}^{\infty} \left[(\psi(v, \bar{v}) - \psi(v, \bar{w})) \frac{\partial v}{\partial \tau} - (\bar{\psi}(w, \bar{w}) - \bar{\psi}(v, \bar{w})) \frac{\partial \bar{v}}{\partial \tau} \right] \mathrm{d}\tau \tag{54}$$

and the path integral measure is defined by

$$\mathcal{D}\mu_\theta(v, \bar{v}) := \lim_{N\to\infty} \prod_{i=1}^{N} \mathrm{d}\mu_{\theta N}(v_{i/N}, \bar{v}_{i/N}). \tag{55}$$

Note that $\psi(z, \bar{z})$ is a canonical conjugation of z, which is defined by

$$\psi(z, \bar{z}) := \frac{\partial K(z, \bar{z})}{\partial z}. \tag{56}$$

As above, the associative symmetrized Berezin star product is defined as the O-star product correctly. The associativity is satisfied as illustrated in figure 1.

4. Construction of Kontsevich star product from multiple star product method

In this section, we show that the multiple Moyal star product corresponds to the path-integral form of the Kontsevich star product on the flat plane. Preparatory to this derivation, we write the multiple Moyal star product in the large-N limit as

$$A_\star(f) := \lim_{N\to\infty} f_{N/N}(x, p) \star f_{N-1/N}(x, p) \star \cdots \star f_{2/N}(x, p) \star f_{1/N}(x, p)$$

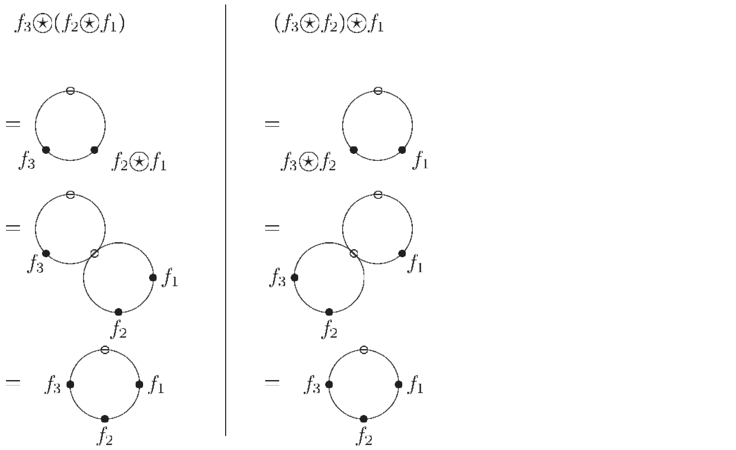

Figure 1. If we denote the O-star product by a circle, the associativity is shown as above. Here ∘ point means the boundary of the O-star product as $(z, \bar{z})$.

$$
\begin{aligned}
&= \lim_{N\to\infty} \int \prod_{i=1}^{N} \frac{\mathrm{d}\xi_{i/N}\,\mathrm{d}\eta_{i/N}}{\pi\hbar N} \frac{\mathrm{d}\xi'_{i/N}\,\mathrm{d}\eta'_{i/N}}{\pi\hbar N} f_{i/N}(\xi_{i/N}, \eta_{i/N}) \\
&\quad \times \exp\left[\frac{2\mathrm{i}}{\hbar N} \sum_{i=1}^{N} \begin{vmatrix} 1 & 1 & 1 \\ \xi'_{i+1/N} & \xi_{i/N} & \xi'_{i/N} \\ \eta'_{i+1/N} & \eta_{i/N} & \eta'_{i/N} \end{vmatrix}\right] \\
&= \int \mathcal{D}\xi\mathcal{D}\eta\mathcal{D}\xi'\mathcal{D}\eta' \prod_{\tau=0}^{1} f(\tau; \xi, \eta) \\
&\quad \times \exp \frac{2\mathrm{i}}{\hbar} \int_0^1 \mathrm{d}\tau \left[\frac{\mathrm{d}\xi'}{\mathrm{d}\tau}(\eta - \eta') - (\xi - \xi')\frac{\mathrm{d}\eta'}{\mathrm{d}\tau}\right]
\end{aligned} \tag{57}
$$

where real fields ξ, η, ξ', η' have boundary conditions

$$
\begin{aligned}
x &= \xi(0) = \xi(1) = \xi'(0) = \xi'(1) \\
p &= \eta(0) = \eta(1) = \eta'(0) = \eta'(1)
\end{aligned} \tag{58}
$$

and functional measures are defined as follows:

$$
\mathcal{D}\xi := \lim_{N\to\infty} \prod_{i=1}^{N} \frac{\mathrm{d}\xi_{i/N}}{\pi \frac{\partial a}{\partial r} N}, \ldots \text{etc.} \tag{59}
$$

In equation (57), we can integrate out ξ', η' by using partial integration and obtain the simplified form

$$
A_\star(f) = \int \mathcal{D}\xi\mathcal{D}\eta \prod_{\tau=0}^{1} f_\tau(\xi(\tau), \eta(\tau)) \exp \frac{\mathrm{i}}{\hbar} \int_0^1 \eta \frac{\partial\xi}{\partial\tau}\,\mathrm{d}\tau. \tag{60}
$$

Here we can change the integration area of τ $(0, 1)$ to $(-\infty, \infty)$ by a reparametrization of τ. Thus equation (60) and the boundary conditions (58) are changed as

$$
A_\star(f) = \int \mathcal{D}\xi\mathcal{D}\eta \prod_{\tau=-\infty}^{\infty} f_\tau(\xi(\tau), \eta(\tau)) \exp \frac{\mathrm{i}}{\hbar} \int_{-\infty}^{\infty} \eta \frac{\partial\xi}{\partial\tau}\,\mathrm{d}\tau, \tag{61}
$$

$$
x = \xi(\pm\infty) = \xi'(\pm\infty) \qquad p = \eta(\pm\infty) = \eta'(\pm\infty). \tag{62}
$$

Next, using (61), we show that the multiple Moyal star product is included in the path-integral form of the Kontsevich star product (27). If we put

$$f_\tau(x,p) = \begin{cases} f(x,p) & (\tau = 1) \\ g(x,p) & (\tau = 0) \\ 1 & (\tau \neq 0,1) \end{cases} \tag{63}$$

then

$$\begin{aligned} f \star g(x,p) &= A_\star(f) \\ &= \lim_{N\to\infty} \cdots \star 1 \star f(x,p) \star 1 \star \cdots \star 1 \star g(x,p) \star 1 \star \cdots \\ &= \int \mathcal{D}\xi\mathcal{D}\eta\ f(\xi(1),\eta(1))g(\xi(0),\eta(0)) \exp\frac{\mathrm{i}}{\hbar}\int_\gamma \mathrm{d}^{-1}\omega_0 \end{aligned} \tag{64}$$

where

$$\mathrm{d}^{-1}\omega_0 := (\mathrm{d}\xi)\eta = \eta\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{65}$$

and

$$\omega_0 = \mathrm{d}(\mathrm{d}^{-1}\omega_0) = \mathrm{d}\xi \wedge \mathrm{d}\eta. \tag{66}$$

Equation (64) corresponds to (27) on the flat plane.

5. Discussions

In this paper, we have proposed a multiple star product method. It is useful for constructing associative star products from *pseudo-associative* star products. We have shown that a *pseudo-associative* S-star product becomes an associative O-star product by using the multiple star product method. This is explained as follows. Although the *pseudo-associative* products break the associativity condition a little, the multiple star product method, which is a set of infinite *pseudo-associative* products, smooths and overcomes this risk and the associativity condition (37) is satisfied. In consequence, the *pseudo-associative* product turns into an associative product within the framework of the path-integral formalism.

The multiple star product method also has been applied to well known associative products such as the Moyal star product. The multiple Moyal star product coincides with the path-integral form of the Kontsevich star product on the flat plane. This result demonstrates the validity of the multiple star product method.

Appendix. Other examples

By using equation (61), we can obtain the transition amplitude in quantum dynamics and the bosonic string generating function. If we put in equation (61)

$$f_\tau(x,p) = \begin{cases} \psi_I(x) & (\tau = t_I) \\ \bar{\psi}_F(x) & (\tau = t_F) \\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(x,p)} & (t_I < \tau < t_F) \\ 1 & (\tau < t_I\ ,\ t_F < \tau) \end{cases} \tag{67}$$

we obtain

$$\begin{aligned} A_\star(f) &= \lim_{N\to\infty} \cdots \star 1 \star \bar{\psi}_F(x) \star \mathrm{e}^{-\frac{\mathrm{i}\varepsilon}{\hbar}H(x,p)} \star \cdots \star \mathrm{e}^{-\frac{\mathrm{i}\varepsilon}{\hbar}H(x,p)} \star \psi_I(x) \star 1 \star \cdots \\ &= \int \mathcal{D}\xi\mathcal{D}\eta\ \bar{\psi}_F(\xi(t_F))\psi_I(\xi(t_I))\mathrm{e}^{\frac{i}{\hbar}\int_{t_I}^{t_F}(\eta\frac{\partial\xi}{\partial\tau} - H(\xi,\eta))\,\mathrm{d}\tau} \\ &= \langle\psi_F, t_F|\psi_I, t_I\rangle \end{aligned} \tag{68}$$

where integration of $\tau < t_I, t_F < \tau$ vanishes because of $f(\tau; x, p) = 1$.

Next, a bosonic string generating function can be derived from infinite-dimensional multiple Moyal star products. In (61), we put

$$f_\tau(x, p) = \mathrm{e}^{\mathrm{i}(k(\tau)x + m(\tau)p)} \tag{69}$$

and obtain

$$\begin{aligned} A_\star(f) &= \lim_{N\to\infty} \mathrm{e}^{\mathrm{i}(k(\infty)x+m(\infty)p)} \star \cdots \star \mathrm{e}^{\mathrm{i}(k(0)x+m(0)p)} \star \cdots \star \mathrm{e}^{\mathrm{i}(k(-\infty)x+m(-\infty)p)} \\ &= \int \mathcal{D}\xi \mathcal{D}\eta \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty}(\eta\frac{\partial\xi}{\partial\tau} + \hbar(k\xi + m\eta))\,\mathrm{d}\tau}. \end{aligned} \tag{70}$$

Here, we generalize $x \to x_{n,\mu}$, $p \to p_n^\mu$ where n runs from 0 to ∞ and μ runs from 1 to dimension d. n can be changed to continuous parameter σ by the following definitions and relation:

$$x_\mu(\sigma) := \sum_{n=0}^{\infty} x_{n,\mu} \cos n\sigma \qquad p^\mu(\sigma) := \sum_{n=0}^{\infty} \frac{p_n^\mu}{n} \sin n\sigma$$

$$\sum_{n=0}^{\infty} x_{n,\mu} p_n^\mu = 2\int_0^{2\pi} x_\mu \frac{\partial p^\mu}{\partial\sigma}\,\mathrm{d}\sigma.$$

Substituting the above relation into A_{mn}, we obtain

$$\begin{aligned} A_\star(f) &= \int \mathcal{D}X \exp\left[\frac{2\mathrm{i}}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}\tau \int_0^{2\pi}\mathrm{d}\sigma\left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\tau} + J^\mu X_\mu\right)\right] \\ X_\mu &= \frac{\xi_\mu + \eta_\mu}{\sqrt{2}} \qquad J^\mu = \hbar\left(\frac{\partial n^\mu}{\partial\sigma} - \frac{\partial m^\mu}{\partial\sigma}\right) \qquad \mathcal{D}X = \mathcal{D}\xi\mathcal{D}\eta. \end{aligned} \tag{71}$$

$A_\star(f)$ becomes the bosonic string generating function. Further details can be found in [SW].

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